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A Contribution to Rational Chebyshev Approximation to Certain Entire Functions in [0, ∞)

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Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function, and set $M(r) = \max_{|z| \leqslant r} |f(z)|$. As usual, $f(z)$ is of order ρ [2, p. 8] if

$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \rho \qquad (0 \leqslant \rho \leqslant \infty).$$

$f(z)$ is of type τ and lower type ω corresponding to the order ρ $(0 < \rho < \infty)$ if

$$\lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix} \tau \\ \omega \end{matrix} \qquad (0 \leqslant \omega \leqslant \tau \leqslant \infty). \tag{1}$$

An entire function $f(z)$ is of perfectly regular growth (ρ, τ) [5, p. 45] if and only if there exist two (finite) positive constants ρ and τ such that

$$\lim_{r \to \infty} \frac{\log M(r)}{r^\rho} = \tau. \tag{2}$$

For an entire function f real and $\neq 0$ on $[0, \infty)$, we set

$$\lambda_{m,n} = \lambda_{m,n}\left(\frac{1}{f}\right) = \min_{r_{m,n} \in \pi_{m,n}} \max_{0 \leqslant x < \infty} \left| r_{m,n}(x) - \frac{1}{f(x)} \right|, \tag{3}$$

where

$$r_{m,n} \equiv \frac{p_m(x)}{q_n(x)}, \qquad p_m(x) \in \pi_m, \qquad q_n(x) \in \pi_n.$$

π_m denote the class of all real polynomials of degree at most m. $\pi_{m,n}$ denote the class of all rational functions of the form $r_{m,n}(x)$.

It is known [5, p. 45] that if $f(z)$ is of type τ and lower type ω, $0 < \omega \leqslant \tau < \infty$, then

$$\limsup_{n \to \infty} \frac{n}{\rho e} | a_n |^{\rho/n} = \tau, \tag{4}$$

$$\liminf_{p \to \infty} \frac{n_p}{\rho e} | a_{n_p} |^{\rho/n_p} \geqslant \omega, \tag{5}$$

for a sequence of numbers n_p satisfying the condition

$$\limsup_{p\to\infty} \frac{n_{p+1}}{n_p} \leqslant \frac{x_1}{x_2}, \tag{6}$$

where x_1 is the largest and x_2 the smallest root of the equation

$$x \log \frac{x}{e} + \frac{\omega}{\tau} = 0. \tag{7}$$

The following theorem relates $\limsup_{n\to\infty} (\lambda_{0,n})^{1/n}$ to the growth of f.

THEOREM A [3, Theorems 2 and 3]. *Let $f(z)$ be an entire function of perfectly regular growth (ρ, τ) with nonnegative coefficients, then*

$$\frac{1}{2^{2+1/\rho}} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{2^{1/\rho}}. \tag{8}$$

In the same direction we have also the following more general result.

THEOREM B [4, Theorem 6]. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $a_0 > 0$ and $a_k \geqslant 0$ for all $k \geqslant 1$, be an entire function of order $\rho(0 < \rho < \infty)$, type τ and lower type ω such that $0 < \omega \leqslant \tau < \infty$. Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} < 1. \tag{9}$$

Whenever Theorem A can be applied, it gives a better upper bound than Theorem B. But Theorem B is valid for a wider class of entire functions.

The aim of this note is to improve, under certain conditions, Theorems A and B.

THEOREM C. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $a_k \geqslant 0$ for all $k \geqslant 0$, is an entire function of order $\rho(0 < \rho < \infty)$, type τ, and lower type ω, with $\tau < 2\omega$, $0 < \omega \leqslant \tau < \infty$, then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\rho x_1}, \tag{10}$$

where x_1 and x_2 are as above.

Remarks. The right inequality (8) follows from (10), because $\tau = \omega$ implies that $x_1 = x_2$. In (10), if ω is very close to τ, then we have a better bound than in (9) for this class of functions. Hence, this theorem is also more general than Theorem B.

Proof of Theorem C. The proof of this theorem is very similar to the proof of Theorem A with one difference. In the proof of Theorem C we sue (4) and (5) instead of (2.4) and (2.5) of [3]. Then, instead of (3.7) of [3] we arrive at

$$\limsup_{p\to\infty} (g_{2n_p-1})^{1/2n_p-1} \leqslant \left(\frac{\tau}{2\omega}\right)^{1/\rho}, \tag{11}$$

where

$$g_n = \sup_{0<x<\infty} \left| \frac{1}{s_n(x)} - \frac{1}{f(x)} \right|, \qquad \forall n \geqslant \tilde{n}.$$

Now from the definition of g_n it is clear that

$$g_m \leqslant g_n, \qquad m \geqslant n \geqslant n^*. \tag{12}$$

For any large n, choose an n_p so that $2n_p - 1 \leqslant n < 2n_{p+1} - 1$. From (12), we have

$$g_n^{1/n} \leqslant g_{2n_p-1}^{1/n} = (g_{2n_p-1}^{1/(2n_p-1)})^{(2n_p-1)/n}.$$

With the restriction $\tau < 2\omega$, it is clear from (11) that $g_{2n_p-1}^{1/(2n_p-1)}$ is less than one for p sufficiently large. Now, replacing n in the exponent of the above expression by $2n_{p+1} - 1$ gives

$$g_n^{1/n} \leqslant (g_{2n_p-1}^{1/(2n_p-1)})^{(2n_p-1)/(2n_{p+1}+1)}. \tag{13}$$

We know from (6) that

$$\limsup_{p\to\infty} \frac{n_{p+1}}{n_p} \leqslant \frac{x_1}{x_2},$$

This inequality along with (11) and (13) gives

$$\limsup_{n\to\infty} (g_n)^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\rho x_1}. \tag{14}$$

By noting that $\lambda_{0,n} \leqslant g_n$ for all large n, (10) follows from (14).

THEOREM D. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geqslant 0$ *for* $k \geqslant 0$, *be an entire function of order* $\rho(0 < \rho < \infty)$, *type* τ *and lower type* ω *with* $\tau < \theta\omega$ *for some* $\theta < 2$, *and* $0 < \omega \leqslant \tau < \infty$. *Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant \left(\frac{\omega}{\tau 2^{2\rho+1}}\right)^{x_1/\rho x_2}. \tag{15}$$

Remarks. For functions of perfectly regular growth, the left inequality (8) follows from (15).

Proof of Theorem D. The coefficients of $f(z)$ being $\geqslant 0$, we have from (2), for all $r \geqslant r_0(\epsilon)$,

$$0 \leqslant f(x) \leqslant f(r) = M(r) \leqslant e^{r^{\rho}\tau(1+\epsilon)}, \qquad 0 \leqslant x \leqslant r, \quad r \geqslant r_0(\epsilon) \quad \text{for} \quad n \geqslant n_0(\epsilon). \tag{16}$$

Now one has from (16)

$$0 \leqslant f(x) \leqslant f\left(\left(\frac{n}{2\tau\rho}\right)^{1/\rho}\right) \leqslant e^{n(1+\epsilon)/2\rho}, \qquad 0 \leqslant x \leqslant \left(\frac{n}{2\tau\rho}\right)^{1/\rho}, \quad n \geqslant n_0(\epsilon). \tag{17}$$

Next, consider $r_{0,n}^* = 1/p_n^*$ from $\pi_{0,n}$ which gives best approximation in the sense of (3), that is

$$\lambda_{0,n} \equiv \min_{p_n^* \in \pi_n} \max \left| \frac{1}{f(x)} - \frac{1}{p_n^*(x)} \right|. \tag{18}$$

From Theorem C it follows that

$$e^{n(1+\epsilon)/2\rho} < (\lambda_{0,n})^{-1} \qquad \text{for all } n \geqslant n(\epsilon). \tag{19}$$

Indeed, if (19) is false, then

$$(\lambda_{0,n}) \geqslant e^{-n(1+\epsilon)/2\rho} \qquad \text{for a sequence of values of } n.$$

Hence, ϵ being arbitrary,

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1-n} \geqslant e^{1/(1/(2\rho))}. \tag{20}$$

But from Theorem C, we have

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\rho x_1}.$$

Equation (20) fails to be true if

$$1/e^{(1/(2\rho))} \geqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\rho x_1}. \tag{21}$$

We can make (21) valid by properly choosing τ and ω.

From (17) we have

$$f(x) < (\lambda_{0,n})^{-1}, \quad 0 \leqslant x \leqslant \left(\frac{n}{2\tau\rho}\right)^{1/\rho}, \quad n \geqslant \max\{n_0(\epsilon), n(\epsilon)\} = \hat{n}. \tag{22}$$

Equation (18) gives, with a little calculation,

$$\frac{-f^2(x)}{f(x) + (\lambda_{0,n})^{-1}} \leqslant p_n^*(x) - f(x) \leqslant \frac{f^2(x)}{(\lambda_{0,n})^{-1} - f(x)}, \tag{23}$$
$$0 \leqslant x \leqslant \left(\frac{n}{2\tau\rho}\right)^{1/\rho}, \qquad n \geqslant \hat{n}.$$

Because the right side of the above inequality is monotonic increasing with x, we can write, from (17),

$$| p_n^*(x) - f(x)| \leqslant \frac{e^{n(1+\epsilon)/\rho}}{(\lambda_{0,n})^{-1} - e^{n(1+\epsilon)/2}}, \quad 0 \leqslant x \leqslant \left(\frac{n}{2\tau\rho}\right)^{1/\rho}, \quad n \geqslant \hat{n}. \tag{24}$$

Now let

$$E_n \equiv \inf_{r_n \in \pi_n'} \sup \left\{| r_n(x) - f(x)|, 0 \leqslant x \leqslant \left(\frac{n}{2\tau\rho}\right)^{1/\rho}\right\}. \tag{25}$$

According to (24), we have

$$E_n \leqslant \frac{e^{n(1+\epsilon)/\rho}}{(\lambda_{0,n})^{-1} - e^{n(1+\epsilon)/2\rho}}, \qquad n \geqslant \hat{n}. \tag{26}$$

To obtain a lower bound for E_n, we use a result of Bernstein [1, p. 10] which gives for the interval $[0, (n/2\tau\rho)^{1/\rho}]$,

$$E_n \geqslant \left(\frac{n}{2\tau\rho}\right)^{(n+1)/\rho} \cdot \frac{f^{(n+1)}(0)}{2^{2n+1}(n+1)!} = \left(\frac{n}{2\tau\rho}\right)^{(n+1)/\rho} \cdot \frac{a_{n+1}}{2^{2n+1}}. \tag{27}$$

Now by (26) and (27) we have

$$\left(\frac{n}{2\tau\rho}\right)^{(n+1)/\rho} \frac{a_{n+1}}{2^{2n+1}} \leqslant \frac{e^{n(1+\epsilon)/\rho}}{(\lambda_{0,n})^{-1} - e^{n(1+\epsilon)/2\rho}}, \qquad n \geqslant \hat{n}. \tag{28}$$

From (5) we have, for a sequence of numbers n_p,

$$a_{n_{p+1}} \geqslant \left(\frac{\rho e\omega(1-\epsilon)}{n_p}\right)^{n_{p+1}/\rho}, \qquad p \geqslant p_1(\epsilon).$$

For this sequence n_p, the left side of (28) is bounded below by

$$\left(\frac{\omega e(1-\epsilon)}{\tau 2^{2\rho+1}}\right)^{n_{p+1}/\rho} \qquad \forall p \geqslant p_1(\epsilon).$$

Consequently, from (28) we have, for all larg $n = n_\rho$,

$$\Gamma\left(\frac{(1-\epsilon)^{1/\rho}\,\omega^{1/\rho}}{e^{\epsilon/\rho}2^{2+(1/\rho)}\tau^{1/\rho}}\right)^n \leqslant \frac{1}{(\lambda_{0,n})^{-1} - e^{n(1+\epsilon)/2\rho}}; \tag{29}$$

where

$$\Gamma = \frac{2(1-\epsilon)^{1/\rho}\,\omega^{1/\rho}}{2^{2+(1/\rho)}\tau^{1/\rho}}.$$

Now (29) holds true for all large n_p only if

$$(\lambda_{0,n_p})^{-1/n_p} \leqslant 2^{2+(1/\rho)}\tau^{1/\rho}e^{\epsilon/\rho}(1-\epsilon)^{-1/\rho}\,\omega^{-1/\rho}. \tag{30}$$

If (30) is not true, then

$$(\lambda_{0,n})^{-1/n} > 2^{2+1/\rho}\tau^{1/\rho}e^{\epsilon/\rho}(1-\epsilon)^{-1/\rho}\omega^{-1/\rho} \qquad \text{for all } n \geqslant n_0.$$

That is

$$\Gamma\lambda_{0,n} < \Gamma\left(\frac{(1-\epsilon)^{1/\rho}\,\omega^{1/\rho}}{2^{2+(1/\rho)}\tau^{1/\rho}e^{\epsilon/\rho}}\right)^n < \frac{1}{(\lambda_{0,n})^{-1} - e^{n(1+\epsilon)/2}} \qquad \text{for all large } n.$$

Therefore,

$$(\lambda_{0,n})^{-1} - e^{n(+\epsilon)/2\rho} < \frac{1}{\Gamma\lambda_{0,n}} \qquad \text{for all large } n. \tag{31}$$

Equation (31) gives, after a simple calculation,

$$\liminf_{n\to\infty}(\lambda_{0,n})^{1/n} > \frac{1}{e^{1/(2\rho)}}, \tag{32}$$

But according to Theorem C,

$$\limsup_{n\to\infty}(\lambda_{0,n})^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\rho x_1}, \tag{33}$$

and

$$\frac{1}{e^{1/(2\rho)}} \geqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\rho x_1}. \tag{21}$$

Hence (32) is false and (30) holds. From (30) we obtain with a little calculation

$$\liminf_{p\to\infty}(\lambda_{0,n_p})^{1/n_p} \geqslant \left(\frac{\omega}{\tau 2^{2\rho+1}}\right)^{1/\rho}.$$

Now following the techniques used at the end of Theorem C, we have

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant \left(\frac{\omega}{\tau 2^{2\rho+1}}\right)^{x_1/\rho x_2}.$$

THEOREM E. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function of order* $\rho(0 < \rho < \infty)$, *type* τ *and lower type* ω, $\tau < 2\omega(0 < \omega \leqslant \tau < \infty)$, *with nonnegative real coefficients. Then*

$$\left(\frac{\omega}{2\tau}\right)^{x_1/\rho x_2} \leqslant \liminf_{n\to\infty} (g_n)^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{1/\rho}, \tag{34}$$

$$\left(\frac{\omega}{2\tau}\right)^{1/\rho} \leqslant \limsup_{n\to\infty} (g_n)^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\rho x_1}. \tag{35}$$

Remark. If $\tau = \omega$, *then as we have observed earlier*, $x_1 = x_2$. Hence, taking in (34) and (35) $\tau = \omega$, we obtain Theorem 1 of [3], that is,

$$\lim_{n\to\infty} (g_n)^{1/n} = \frac{1}{2^{1/\rho}}. \tag{36}$$

Proof of Theorem E. From the hypothesis we have

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} = \frac{\sum_{k=n+1}^{\infty} a_k x^k}{f(x)\, s_n(x)} \geqslant \frac{a_{n+1} x^{n+1}}{f^2(x)}, \qquad \forall x > 0, \qquad \forall n \geqslant n^*.$$

With $n + 1 = n_p$, we have from (5),

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} > \left\{\frac{(\rho\omega - \epsilon)^{n+1}}{(n+1)!}\right\}^{1/\rho} \cdot \frac{x^{n+1}}{f^2(x)} \qquad \forall x > 0, \qquad p \geqslant p_1(\epsilon).$$

We know from (16) that there exists an $R_1(\epsilon) > 0$ such that

$$f(x) < e^{(\tau+(\epsilon/\rho))x^\rho} \qquad \forall x > R_1(\epsilon).$$

Therefore,

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} > \left(\frac{(\rho\omega - \epsilon)^{n+1}}{(n+1)!}\right)^{1/\rho} \cdot \frac{x^{n+1}}{e^{2(\tau+(\epsilon/\rho))x^\rho}}, \tag{37}$$

$$n + 1 = n_p, \qquad p \geqslant p_1(\epsilon), \qquad x > R_1(\epsilon).$$

With $x^\rho = (n+1)/(2(\rho\tau + \epsilon))$, which is compatable with $x > R_1(\epsilon)$ if n is large, we obtain from (37):

$$g_n > \left\{\frac{(\rho\omega - \epsilon)^{n+1}}{(n+1)!}\right\}^{1/\rho} \cdot \left\{\frac{n+1}{2(\rho\tau + \epsilon)}\right\}^{(n+1)/\rho} e^{-(n+1)/\rho}.$$

Hence, it readily follows that

$$\liminf_{p\to\infty} (g_{n_p-1})^{1/n_p-1} \geqslant \left(\frac{\omega}{\tau 2}\right)^{1/\rho}. \tag{38}$$

Using the same technique that established (14) from (11), we obtain from (38)

$$\liminf_{n\to\infty} (g_n)^{1/n} > \left(\frac{\omega}{2\tau}\right)^{x_1/x_2\rho}. \tag{39}$$

Now (34) follows from (11) and (39), and (35) follows from 14) and (38).

Some remarks on Theorems C and D. For functions of perfectly regular growth (ρ, τ) we have from Theorems C and D

$$\frac{1}{2^{2+(1/\rho)}} \leqslant \liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{2^{1/\rho}}. \tag{40}$$

Equation (40) suggests that the limit may exist for a certain class of entire functions; (40) also suggests that one cannot replace 2 on the right by any number greater than 2 for any ρ.

THEOREM F. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 > 0, a_k \geqslant 0$ for all $k \geqslant 1)$ be an entire function of order $\rho(0 < \rho < \infty)$, type τ and lower type ω, such that $0 < \omega \leqslant \tau < \infty$. Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{e^{\omega/(2\rho e\tau)}}, \qquad \text{if} \quad \omega \leqslant 2\rho e\tau \log[4(2^{1/\rho} - 1)], \tag{41}$$

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{4(2^{1/\rho} - 1)}, \qquad \text{if} \quad \omega \geqslant 2\rho e\tau \log[4(2^{1/\rho} - 1)]. \tag{42}$$

Proof. For each $r > 0$, let $q_n(x; r) \in \pi_n$ denote the best Chebyshev approximation to f in $[0, r]$ so that

$$\|f - q_n(\cdot; r)\|_{[0;r]} = \inf_{\sigma_n \in \pi_n} \|f - \sigma_n\|_{[0;r]} = \sigma_n(r). \tag{43}$$

It is known that there exist points $0 < x_1(r) < x_2(r) < \cdots < x_{n+1}(r) < r$ such that $q_n(x_j(r); r) = f(x_j(r))$, $1 \leqslant j \leqslant n + 1$. By expressing $q_n(x; r)$ as a Newton interpolation series, we have

$$q_n(x; r) = f(x_1(r)) + f[x_1(r), x_2(r)](x - x_1(r)) + \cdots$$
$$+ f[x_1(r),\ldots, x_{n+1}(r)] \cdot \prod_{j=1}^{n} (x - x_j(r)),$$

where $f(x_1(r),..., x_{j+1}(r)]$ is the devided difference of f at the points $x_1(r),..., x_{j+1}(r)$. It is known that

$$f[x_1(r),..., x_{j+1}(r)] = (f^{(j)}(\xi))/j!, \qquad \text{where} \quad x_1(r) < \xi < x_{j+1}(r).$$

Since the a_k are all $\geqslant 0$, the same holds for the divided differences. So $q_n(x; r)$ is monotonically increasing as a function of x for all $x \geqslant r$. Now let

$$P_n(x; r) \equiv q_n(x; r) + \delta_n(r), \qquad \text{for each } n \geqslant 0.$$

From (43), it is evident that

$$P_n(x; r) \geqslant f(x) \geqslant f(0) > 0, \qquad \text{for all } x \in [0, r]. \tag{44}$$

Moreover, from the monotone nature of $P_n(x; r)$ as a function of x for all $x \geqslant r$, we also have that

$$P_n(x; r) \geqslant f(r) \qquad \text{for all } x \geqslant r.$$

But

$$f(x) \geqslant f(r) > 0 \qquad \text{for all } x \geqslant r.$$

Therefore, from the above two inequalities,

$$\left| \frac{1}{f(x)} - \frac{1}{P_n(x; r)} \right| \leqslant \frac{2}{f(r)} \qquad \text{for all } x \geqslant r. \tag{45}$$

On the other hand, it follows from (43) that

$$\left| \frac{1}{f(x)} - \frac{1}{P_n(x; r)} \right| = \left| \frac{P_n(x; r) - f(x)}{f(x)\, P_n(x; r)} \right| \leqslant \frac{2\delta_n(r)}{f^2(0)}, \qquad x \in [0, r]. \tag{46}$$

Now by using again a result of S. N. Bernstein [1, p. 10] over the interval $[0, r]$, along with the hypothesis that the Taylor coefficients are $\geqslant 0$, we have

$$\delta_n(r) \leqslant \frac{f^{(n+1)}(r)\, r^{n+1}}{(n+1)!\, 2^{2n+1}}.$$

That is,

$$\delta_n(r) \leqslant \frac{r^{n+1}}{2^{2n+1}} \sum_{j=1}^{\infty} \frac{a_{n+1+j}(n+1+j)!\, r^j}{(n+1)!(j)!}. \tag{47}$$

Now from (4) we have, for sufficiently large n,

$$a_{n+j+1} \leqslant \left\{ \frac{\rho e(\tau + \epsilon)}{n + j + 1} \right\}^{(n+j+1)/\rho}, \qquad \text{for} \quad j = 0, 1, 2, 3,....$$

By substituting this in (47), we obtain

$$\delta_n(r) \leqslant \frac{r^{n+1}}{2^{2n+1}} \sum^{\infty} \frac{a_{n+1+j}(n+1+j)!\, r^j}{(n+1)!(j)!}$$

$$\leqslant \frac{r^{n+1}}{2^{2n+1}} \sum_{j=0}^{\infty} \left(\frac{\rho e(\tau+\epsilon)}{n+1+j}\right)^{(n+1+j)/\rho} r^j \frac{(n+1+j)!}{(n+1)!(j)!}$$

$$\leqslant \left\{r\left(\frac{\rho e(\tau+\epsilon)}{(n+1)}\right)^{1/\rho}\right\}^{n+1} \frac{1}{2^{2n+1}} \sum_{j=0}^{\infty} \left\{\left(\frac{\rho e(\tau+\epsilon)}{n+1}\right)^{1/\rho} r\right\}^j \frac{(n+1+j)!}{(n+1)!(j)!}.$$

Now choose

$$r\left(\frac{\rho e(\tau+\epsilon)}{(n+1)}\right)^{1/\rho} = \frac{1}{2^{1/\rho}}. \tag{48}$$

Then

$$\delta_n(r) \leqslant 2^{(n+1)\rho^{-1}+2n+1} \sum_{j=0}^{\infty} \frac{1}{2^{j/\rho}} \frac{(n+1+j)!}{(n+1)!(j)!}$$

$$= 2^{(n+1)\rho^{-1}+2n+1} \left(\frac{2^{1/\rho}}{2^{1/\rho}-1}\right)^{n+2},$$

that is

$$\delta_n(r) \leqslant \frac{2^{1/\rho}}{2^{2n+1}(2^{1/\rho}-1)^{n+2}},$$

so that,

$$\left|\frac{1}{f(x)} - \frac{1}{P_n(x;r)}\right| \leqslant \frac{2\delta_n(r)}{a_0^2} \leqslant \frac{2^{1/\rho}}{2^{2n}(2^{1/\rho}-1)^{n+2}\, a_0^2}, \qquad x \in [0, r]. \tag{49}$$

From the definition of lower type we have $f(r) \geqslant e^{r^{\rho}(\omega-\epsilon)}$, $r \geqslant r_\epsilon$.

Now substituting in this inequality the value of r from (48), we have

$$f(r) \geqslant \exp\left(\frac{(n+1)(\omega-\epsilon)}{2\rho e(\tau+\epsilon)}\right), \qquad r \geqslant r_\epsilon.$$

From (45) and this inequality,

$$\frac{2}{f(r)} \leqslant \frac{2}{\exp\left(\dfrac{(n+1)(\omega-\epsilon)}{2\rho e(\tau+\epsilon)}\right)}.$$

Hence

$$\left|\frac{1}{f(x)} - \frac{1}{P_n(x; r)}\right| \leqslant \frac{2}{f(r)} \leqslant \frac{2}{\exp\left(\dfrac{(n+1)(\omega-\epsilon)}{2\rho e(\tau+\epsilon)}\right)} \qquad \forall x \geqslant r. \quad (50)$$

Now if we set $P_n(x) = P_n(x; r) = P_n(x; r(n))$, we have from (49) and (50), ϵ being arbitrary,

$$\limsup_{n\to\infty} \left|\frac{1}{f(x)} - \frac{1}{P_n(x)}\right|^{1/n} \leqslant \max\left\{\frac{1}{4(2^{1/\rho}-1)}, \frac{1}{e^{\omega/2\rho e\tau}}\right\}. \quad (51)$$

If $\omega \leqslant \tau 2\rho e \log[4(2^{1/\rho} - 1)]$, then clearly (41) follows, while if

$$\omega > \tau 2\rho e \log[4(2^{1/\rho} - 1)],$$

then (42) is valid.

Remark on Theorem F. Theorem F also improves Theorem A. Unlike Theorem C, this theorem strongly depend on the order ρ of the function via (41) and (42).

THEOREM G. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 > 0,\ a_k \geqslant 0,\ k = 1, 2,...)$ *be an entire function of order* $\rho(0 < \rho < \infty)$, *type* τ *and lower type* $\omega(0 < \omega \leqslant \tau < \infty)$. *Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{e^{\omega/[\rho e\tau(1+2^{1/\rho})^{\rho}]}}. \quad (52)$$

Proof. The proof proceeds along the same lines as that of Theorem F, except that we choose here instead of (48),

$$\left(\frac{\rho e(\tau+\epsilon)}{(n+1)}\right)^{1/\rho} = \frac{1}{2^{1/\rho}+1}. \quad (48')$$

Then we obtain instead of (51),

$$\limsup_{n\to\infty} \left\|\frac{1}{f(x)} - \frac{1}{P_n(x)}\right\|^{1/n} \leqslant \max\left(\frac{1}{2^{2+1/\rho}}, \frac{1}{e^{\omega/\rho e\tau(1+2^{1/\rho})^{\rho}}}\right). \quad (51')$$

Here, obviously, $1/2^{2+1/\rho} \leqslant 1/e^{\omega/\rho e\tau(1+2^{1/\rho})^{\rho}}$ for any $\rho(0 < \rho < \infty)$, τ and $\omega(0 < \omega \leqslant \tau < \infty)$. Hence (52) follows.

Remarks on Theorem G. This theorem includes Theorem B. Theorems C, F and G suggest that by restricting the growth of the function, one can get better upper bounds.

Note added in proof. Since this paper was submitted for publication, much progress has been made in several directions; the interested reader may refer to Refs. 6–11.

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